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DYNAMICS OF CORIOLIS VIBRATORY GYROSCOPES IN CONTROL SYSTEMS

Analysis of the Coriolis vibratory gyroscopes sensitive element dynamics in terms of the amplitude-phase variables led to the proper transfer functions of such inertial sensors is proposed in this paper. Obtained transfer functions are then analysed and simplified for the several special cases. Performance of the simplified transfer functions is also analysed compared to the accurate numerical model of the sensitive elements dynamics. Obtained transfer functions enable study of the Coriolis vibratory gyroscopes as elements of control systems.

Keywords: *Coriolis vibratory gyroscope, dynamics, control systems.*

Introduction

Coriolis vibratory gyroscopes (CVGs) received significant amount of interest from the both scientific and engineering communities due to the possibility to fabricate sensitive elements of such gyroscopes in miniature form by using modern microelectronic mass-production technologies. Such gyroscopes are frequently referred to as MEMS (Micro-Electro-Mechanical-Systems) gyroscopes. Being based on sensing of Coriolis acceleration due to the rotation in oscillating structures, CVGs have a lot more complicated mathematical models, comparing to the conventional types of gyroscopes. One of such complication is a result of the useful signal proportional to the external angular rate being modulated with the intentionally excited primary oscillations [1 – 3]. From the mathematical modelling point of view, this leads to necessity to "demodulate" the solution in terms of the sensitive element displacements to obtain practically feasible insights into CVG dynamics and errors.

Current state analysis. From the control systems point of view, conventional representation of CVGs incorporates primary oscillation excitation signal as an input to the dynamic system, and unknown angular rate as a coefficients of its transfer functions [3]. As a result, dynamics of CVGs has been analysed mainly in steady state, while transient process analysis has been omitted due to its apparent complexity.

This paper describes new method of CVG dynamics analysis by means of complex amplitude-phase variables, which enables having angular rate as an input to the dynamic system.

Problem formulation. In the most generalized form, motion equations of the CVG sensitive element both with translational and rotational motion could be represented in the following form:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1 k_1 \dot{x}_1 + (k_1^2 - d_1\Omega^2)x_1 + g_1\Omega\dot{x}_2 + d_3\dot{\Omega}x_2 = q_1(t); \\ \ddot{x}_2 + 2\zeta_2 k_2 \dot{x}_2 + (k_2^2 - d_2\Omega^2)x_2 - g_2\Omega\dot{x}_1 - \dot{\Omega}x_1 = q_2(t). \end{cases} \quad (1)$$

Here x_1 and x_2 are the generalized coordinates that describe primary (excited) and secondary (sensed) motions of the sensitive element respectively, k_1 and k_2 are the corresponding natural frequencies, ζ_1 and ζ_2 are the dimensionless relative damping coefficients, Ω is the measured angular rate, which is orthogonal to the axes of primary and secondary motions, q_1 and q_2 are the generalized accelerations due to the external forces acting on the sensitive element. The remaining dimensionless coefficients are different for the sensitive elements exploiting either translational or rotational motion. For the translational sensitive element they are $d_1 = d_2 = 1$, $d_3 = m_2/(m_1 + m_2)$, $g_1 = 2m_2/(m_1 + m_2)$, $g_2 = 2$, where were m_1 and m_2 are the masses of the outer frame and the internal massive element. In case of the rotational motion of the sensitive element, these coefficients are the functions of different moments of inertia (for greater details see [4]).

In the presented above motion equations, the angular rate is included as an unknown and variable coefficient rather than an input to the double oscillator system. Conventional control systems representation of such a dynamic system is shown in Fig. 1.

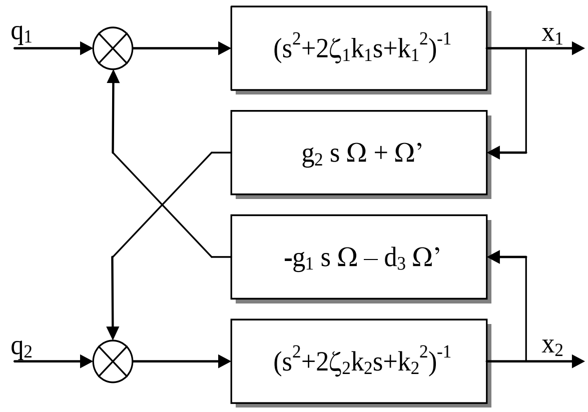

Fig. 1. Conventional representation of CVG in control systems

In order to identify the angular rate one must detect secondary oscillations of the sensitive element and measure its amplitude, which is approximately directly proportional to the angular rate, and phase, which gives the sign.

Compatible with the most control problems CVG dynamics representation should have the unknown angular rate as an input and its measured value as an output.

Main section

Motion equations simplifications. In order to make the equations (1) suitable for to the transient process analysis we must make the following assumptions: angular rate is small comparing to the primary and secondary natural frequencies so that

$$k_1^2 >> d_1\Omega^2;\ k_2^2 >> d_2\Omega^2, \tag{2}$$

and rotational and Coriolis accelerations acting along primary oscillation axis are negligible in comparison to the accelerations from driving forces

$$g_1\Omega\dot{x}_2 + d_3\dot{\Omega}x_2 << q_1(t). \tag{3}$$

Taking into considerations assumptions (2) and (3), motions equations (1) could be simplified to the following form:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1 k_1 \dot{x}_1 + k_1^2 x_1 = q_1(t); \\ \ddot{x}_2 + 2\zeta_2 k_2 \dot{x}_2 + k_2^2 x_2 = g_2\Omega\dot{x}_1 + \dot{\Omega}x_1. \end{cases} \tag{4}$$

Here we also assumed that no external driving forces are affecting the secondary oscillations, which means that $q_2(t) = 0$. System of equations (4) is now perfectly suitable for further transformations towards the desired representation in terms of the unknown angular rate.

Amplitude-phase motion equations. As has been shown in [5], by means of a proper chosen phase shift of the excitation voltage applied to the sensitive element, the excitation force could be shaped to the perfect harmonic form. Using exponential representation of complex numbers, such a driving force $q_1(t)$ could be represented as

$$q_1(t) = q_{10}\sin(\omega t) = \mathrm{Im}\{q_{10}e^{j\omega t}\}. \tag{5}$$

Here ω is the excitation frequency given in radians per second, q_{10} is the constant excitation acceleration amplitude.

Non-homogeneous solutions of the motion equations (1) or (4) for primary and secondary oscillations are searched in a similar form

$$x_1(t) = \mathrm{Im}\{A_1(t)e^{j\omega t}\};\ A_1(t) = A_{10}(t)e^{j\varphi_{10}(t)};$$
$$x_2(t) = \mathrm{Im}\{A_2(t)e^{j\omega t}\};\ A_2(t) = A_{20}(t)e^{j\varphi_{20}(t)}, \tag{6}$$

where A_{10} and A_{20} are the primary and secondary oscillation amplitudes, φ_{10} and φ_{20} are the corresponding phase shifts relatively to the excitation force. Although these quantities are real (non-imaginary), they are combined in complex amplitude-phase variables A_1 and A_2.

Substituting expressions (5) and (6) into equations (4) results in the following motions equations in terms of the complex amplitude-phase variables rather than real generalized coordinates:

$$\begin{cases} \ddot{A}_1 + 2(\zeta_1 k_1 + j\omega)\dot{A}_1 + \\ +(k_1^2 - \omega^2 + 2j\omega k_1\zeta_1)A_1 = q_{10}; \\ \ddot{A}_2 + 2(\zeta_2 k_2 + j\omega)\dot{A}_2 + (k_2^2 - \omega^2 + \\ +2j\omega k_2\zeta_2)A_2 = (j\omega g_2\Omega + \dot{\Omega})A_1 + g_2\dot{A}_1\Omega. \end{cases} \tag{7}$$

Equations (7) describe variations of the amplitude and phase of the primary and secondary equations in time with respect to the unknown non-constant angular rate $\Omega(t)$. This allows conducting analysis of the Coriolis vibratory gyroscope dynamics without constraining the angular rate to be constant or slowly varying.

Analysing system (7), one can see that the first equation can be solved separately from the second one. After homogeneous solutions of the first equation faded out, only non-homogenous solution remains. In this case, amplitude of the primary oscillations is

$$A_1 = \frac{q_{10}}{k_1^2 - \omega^2 + 2jk_1\zeta_1\omega}, \tag{8}$$

and it is constant in time, yielding $\ddot{A}_1 = \dot{A}_1 = 0$. Indeed, most of the time measurements of the angular rate are performed when primary oscillations have already settled. As a result, only equation for the secondary oscillations remains, in which the complex primary amplitude A_1 is just a constant parameter given by (8):

$$\begin{aligned} &\ddot{A}_2 + 2(\zeta_2 k_2 + j\omega)\dot{A}_2 + (k_2^2 - \omega^2 + \\ &+2j\omega k_2\zeta_2)A_2 = (j\omega g_2\Omega + \dot{\Omega})A_1. \end{aligned} \tag{9}$$

Equation (7) now describes amplitude-phase of the secondary oscillations with respect to the settled primary oscillations.

System transfer functions. Having CVG sensitive element motion equation in the form (9), allows analysis of its transient processes in amplitudes and phases with respect to arbitrary angular rates applied to the system. Application of the Laplace transformation to the equations (7) with respect to zero initial conditions for all time-dependent variables results in the following expressions:

$$\begin{aligned} &[(s + j\omega)^2 + 2\zeta_2 k_2(s + j\omega) + k_2^2]A_2(s) = \\ &= A_1[s + jg_2\omega]\Omega(s). \end{aligned} \tag{10}$$

Solution of the algebraic equation (10) for the secondary amplitude-phase Laplace transform is

$$A_2 = \frac{A_1 \cdot (s + jg_2\omega)}{(s + j\omega)^2 + 2\zeta_2 k_2 (s + j\omega) + k_2^2}\Omega. \quad (11)$$

Considering the angular rate as an input, the system transfer function for the secondary amplitude-phase is

$$W_2(s) = \frac{A_2(s)}{\Omega(s)} = \frac{A_1(s + jg_2\omega)}{(s + j\omega)^2 + 2\zeta_2 k_2 (s + j\omega) + k_2^2} = \frac{q_{10}(s + jg_2\omega)}{[(s + j\omega)^2 + 2\zeta_2 k_2 (s + j\omega) + k_2^2]} \times \frac{1}{[k_1^2 - \omega^2 + 2j\omega k_1 \zeta_1]}. \quad (12)$$

One should note that transfer function (12) has complex coefficients, which results in the complex system outputs as well. Although it is somewhat unusual, it still enables us to analyse CVG dynamics and transient processes due to the angular rate in an open-loop dynamic system.

Amplitude and phase responses. In order to calculate the amplitude response of the system using transfer function (12), Laplace variable s must be replaced with the Fourier variable $j\lambda$, where λ is the frequency of the angular rate oscillations:

$$W_2(j\lambda) = \frac{jq_{10}(\lambda + g_2\omega)}{[k_2^2 - (\lambda + \omega)^2 + 2j\zeta_2 k_2 (\lambda + \omega)]} \times \frac{1}{[k_1^2 - \omega^2 + 2j\omega\zeta_1 k_1]}. \quad (13)$$

Absolute value of the complex function (13) is the amplitude response of the secondary oscillations amplitude to the harmonic angular rate, and the corresponding phase of the complex function is the phase response [5]:

$$A(\lambda) = \frac{q_{10}(\lambda + g_2\omega)}{[(k_2^2 - (\lambda + \omega)^2)^2 + 4\zeta_2^2 k_2^2 (\lambda + \omega)^2]^{\frac{1}{2}}} \times \frac{1}{[(k_1^2 - \omega^2)^2 + 4\zeta_1^2 k_1^2 \omega^2]^{\frac{1}{2}}};$$

$$\varphi(\lambda) = \tan^{-1}\{\Delta^{-1}[k_2^2 - (\lambda + \omega)^2][k_1^2 - \omega^2] - \Delta^{-1} 4k_1 k_2 \zeta_1 \zeta_2 \omega(\lambda + \omega); \quad (14)$$

$$\Delta = 2[k_2\zeta_2(\lambda + \omega)(k_1^2 - \omega^2) + k_1\zeta_1\omega(k_2^2 - (\lambda + \omega)^2)].$$

One should note that, assuming constant angular rate ($\lambda = 0$) in the expressions (12) the well known expressions ([4]) for the amplitude and phase of the secondary oscillations is obtained.

Analysis of the expressions (12) shows that effect from the oscillating angular rate is practically equivalent to shift of the excitation frequency by the frequency of the angular rate. This causes CVGs, especially those with high Q-factor, to loose its resonant tuning, which in turn results in significant variation of its scale factor (dynamic error). Solution of this problem by means of proper choice of natural frequency split and damping has been suggested in [6].

System poles and stability. Both stability and unit-step transient process quality depend on position of the system poles in the real-imaginary plane. Poles of the transfer function (12) are as follows:

$$s_{1,2} = -k_2\zeta_2 \pm jk_2\sqrt{1 - \zeta_2^2} - j\omega. \quad (15)$$

Analysing expressions (15), it is easy to see that CVGs are inherently stable.

Indeed, if the relative damping coefficient $\zeta_2 \le 1$, then real parts of the poles are

$$-k_2\zeta_2 < 0.$$

If the relative damping coefficient $\zeta_2 > 1$, then real parts are

$$-k_2(\zeta_2 \pm \sqrt{\zeta_1^2 - 1}) < 0.$$

Ideal (half-oscillatory) unit-step angular rate transient process in secondary oscillations amplitude is achievable if imaginary parts of the poles (15) are zero.

One pole has large imaginary part

$$-k_2\sqrt{1 - \zeta_2^2} - \omega < 0,$$

which is always way below zero, and corresponds to high frequency oscillations in the envelope.

The second pole is responsible for the low frequency oscillations, and is the most essential for the transient process.

Case of slowly varying amplitudes. Another consequence of the presented above analysis of the system poles is that actual amplitude of the secondary oscillations is mainly defined by the low frequency pole, while effect from the high frequency pole can be neglected, since it will be removed during demodulation process. In other words, predominant behaviour is a slow variation of the amplitude and phase. Neglecting the second order derivative in the equation (9) yields

$$2(\zeta_2 k_2 + j\omega)\dot{A}_2 + (k_2^2 - \omega^2 + 2j\omega k_2\zeta_2)A_2 = (j\omega g_2\Omega + \dot{\Omega})A_1,$$

and the corresponding angular rate transfer function becomes

$$W_2 = \\ = \frac{q_{10}(s + jg_2\omega)}{[2\zeta_2 k_2 s + k_2^2 - \omega^2 + j2\omega(\zeta_2 k_2 + s)]} \times \\ \times \frac{1}{[k_1^2 - \omega^2 + 2j\omega k_1\zeta_1]}. \quad (16)$$

Complex transfer function (16) is simpler in comparison to the function (12) and could replace it in certain specific problems when slow oscillations analysis is required.

Real and imaginary transfer functions. While simulating dynamics of CVG based on the complex amplitude-phase transfer functions (12) or (16) one could have problems dealing with complex coefficients of these transfer functions.

One way to avoid this problem is to consider real and imaginary parts of complex amplitude as separate signals, which are then combined together to produce real amplitude and phase.

In order to obtain transfer functions for such signals let us represent primary and secondary amplitudes as:

$$A_1 = A_{1R} + jA_{1I}; \; A_2 = A_{2R} + jA_{2I}. \quad (17)$$

Primary oscillations components can be easily found by means of substituting expressions (17) into formula (8) thus yielding

$$A_{1R} = \frac{q_{10}(k_1^2 - \omega^2)}{(k_1^2 - \omega^2)^2 + 4k_1^2\zeta_1^2\omega^2}; \\ A_{1I} = -\frac{2q_{10} j\omega k_1\zeta_1}{(k_1^2 - \omega^2)^2 + 4k_1^2\zeta_1^2\omega^2}. \quad (18)$$

At the same time, substituting expressions (18) into the motion equation (9), and applying Laplace transformation with zero initial conditions gives

$$\begin{cases} (k_2^2 - \omega^2 + 2k_2\zeta_2 s + s^2)A_{2R}(s) - \\ -2\omega(k_2\zeta_2 + s)A_{2I}(s) = \\ = (A_{1R}s - A_{1I}g_2\omega)\Omega(s); \\ (k_2^2 - \omega^2 + 2k_2\zeta_2 s + s^2)A_{2I}(s) + \\ +2\omega(k_2\zeta_2 + s)A_{2R}(s) = \\ = (A_{1I}s + A_{1R}g_2\omega)\Omega(s). \end{cases} \quad (19)$$

Resolving algebraic system (19) with respect to unknown real and imaginary parts of the secondary complex amplitude results in

$$A_{2R}(s) = \frac{A_{1R}M_{RR}(s) + A_{1I}M_{RI}(s)}{P(s)}\Omega(s); \\ A_{2I}(s) = \frac{A_{1R}M_{IR}(s) + A_{1I}M_{II}(s)}{P(s)}\Omega(s). \quad (20)$$

Here the numerator polynomials from the real and imaginary parts of primary amplitudes are given by the following expressions:

$$M_{RR}(s) = \\ = s(k_2^2 + 2k_2\zeta_2 s + s^2) - \omega^2(s - 2g_2(s + k_2\zeta_2)); \\ M_{RI}(s) = \\ = \omega[2s(s + k_2\zeta_2) - g_2(k_2^2 - \omega^2 + 2k_2\zeta_2 s + s^2)]; \\ M_{II}(s) = \\ = 2\omega^2 g_2(s + k_2\zeta_2) + s(k_2^2 - \omega^2 + 2k_2\zeta_2 s + s^2)]; \\ M_{IR}(s) = \\ = \omega[g_2(k_2^2 - \omega^2 + 2k_2\zeta_2 s + s^2) - 2s(s + k_2\zeta_2)]; \\ P(s) = \\ = 4(s + k_2\zeta_2)^2\omega^2 + (k_2^2 - \omega^2 + 2k_2\zeta_2 s + s^2)^2. \quad (21)$$

Obtained expressions (18), (20), and (21) allow analysis of CVG dynamics in control system without necessity to involve complex-valued signals.

Simplified transfer function and its accuracy. There is quite an important special case, when complex transfer functions transform to the simple real-valued one. Assuming equal primary and secondary natural frequencies ($k_1 = k_2 = k$), equal damping ratios ($\zeta_1 = \zeta_2 = \zeta$), resonance excitation ($\omega = k$), and constant angular rate, one can easily obtain

$$A_{20}(s) = \sqrt{A_{2R}^2(s) + A_{2I}^2(s)} = \\ = \frac{q_{10}g_2}{4k^2\zeta(s + k\zeta)}\Omega(s). \quad (22)$$

In this case, secondary amplitude (22) is related to the input angular rate by means of the following transfer function:

$$W_{20}(s) = \frac{A_{20}(s)}{\Omega(s)} = \frac{q_{10}g_2}{4k^2\zeta(s + k\zeta)}. \quad (23)$$

As one can see, the simplified CVG transfer function (23) describes a simple first-order system with exponential transient process. Needless to say, that possibility to use function (23) for “non-tuned” CVG as well is highly desired. Therefore let us evaluate accuracy of the function (23) in representing general case of CVG dynamics. In order to do that, let us compare transient processes produced by the simplified transfer function and by a numerical solution of the equations (1) with subsequent demodulation. As a performance criterion the following integral function is used:

$$J(\delta k, \delta\zeta) = \int_0^T [A_{20}(t) - A_{20}^*(t)]^2 dt. \quad (24)$$

Here $\delta k = k_2 / k_1$ is the ratio of the natural frequencies, $\delta\zeta = \zeta_2 / \zeta_1$ is the ratio of the relative damping ratios, $A_{20}^*(t)$ is the demodulated secondary amplitude produced by the “realistic” model.

Graphic plot of the functional (24) is shown below in the fig. 2.

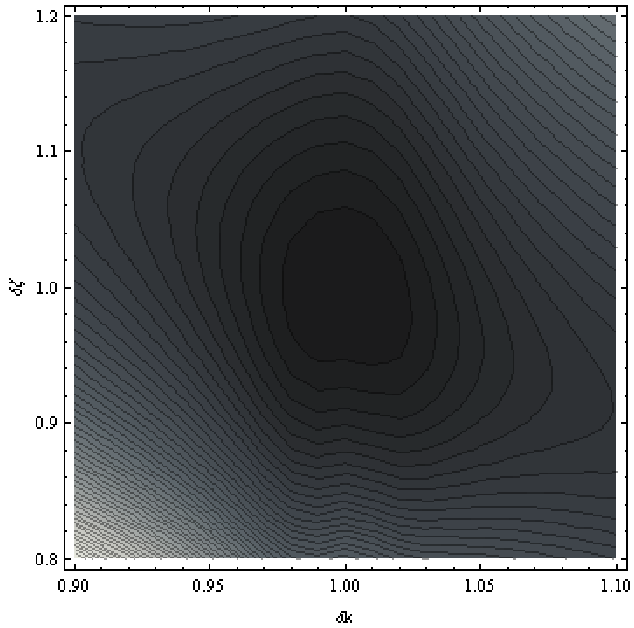

Fig. 2. Integral error of transient process representation

In this figure darker colours correspond to the zero error of the transient process representation. Here the central dark spot corresponds to the perfectly tuned device.

One can see, that wide range of sensitive elements with varying ratio of the natural frequencies and ratio of relative damping still could be represented by the transfer function (23) with acceptably low integral error.

Conclusions

Presented above analysis of CVG dynamics using amplitude-phase complex variables resulted in obtaining system transfer functions, where measured angular rate became an input rather than a parameter. This makes possible to analyse amplitude and phase responses of CVG, its transient processes in already demodulated signals, optimise transient process characteristics. Excellent performance of the obtained simplified transfer functions has been demonstrated using numerical analysis of the integral error analysis.

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ДИНАМІКА КОРІОЛІСОВИХ ВІБРАЦІЙНИХ ГІРОСКОПІВ В СИСТЕМАХ КЕРУВАННЯ

В.О. Апостолюк

В цій статті запропоновано аналіз динаміки чутливого елемента коріолісових вібраційних гіроскопів в амплітудно-фазових змінних, що веде до виведення коректної передатної функції таких інерціальних датчиків. Отримана передатна функція проаналізована та спрощена для декількох важливих часткових випадків. Також було проаналізовано якість спрощеної передатної функції порівняно із точною чисельною моделлю. Отримана передатна функція дозволяє вивчення коріолісових вібраційних гіроскопів як елементів систем керування.

Ключові слова: *коріолісовий вібраційний гіроскоп, динаміка, контрольні системи.*

ДИНАМИКА КОРИОЛИСОВЫХ ВИБРАЦИОННЫХ ГИРОСКОПОВ В СИСТЕМАХ УПРАВЛЕНИЯ

В.А. Апостолюк

В этой статье предложен анализ динамики чувствительного элемента кориолисовых вибрационных гироскопов в амплитудно-фазовых переменных, который позволяет получить корректную передаточную функцию таких инерциальных датчиков. Полученная передаточная функция проанализирована и упрощена для нескольких важных частных случаев. Также было проанализировано качество упрощенной передаточной функции по сравнению с точной численной моделью. Полученная передаточная функция позволяет изучать кориолисовы вибрационные гироскопы как элементы систем управления.

Ключевые слова: *кориолисовый вибрационный гироскоп, динамика, контрольные системы.*