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SYNTHESIS OF COMPENSATED CORIOLIS VIBRATORY GYROSCOPES

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Synthesis of the Coriolis vibratory gyroscopes compensation with the help of the Wiener-Kolmogorov procedure led to the proper transfer function of the optimal feedback controller is proposed in this paper. CVG as the object of compensation is considered as sensitive element transfer function, which was obtained after analysis of it's dynamics in terms of the amplitude-phase variables. Efficiency of the obtained transfer function of the optimal feedback controller is demonstrated by means of numerical simulation.

Описано синтез передавальної функції оптимального регулятора зворотного зв'язку за допомогою підходу Вінера–Колмогорова. Коріолісів вібраційний гіроскоп як об'єкт компенсації розглянуто як передавальну функцію його чутливого елемента, яка була отримана після аналізу його динаміки у виразах амплітудно-фазових змінних. Ефективність отриманої передавальної функції оптимального регулятора зворотного зв'язку продемонстровано за допомогою числового моделювання.

Описан синтез передаточной функции оптимального регулятора обратной связи с использованием подхода Винера–Колмогорова. Кориолисов вибрационный гироскоп как объект компенсации рассмотрен в качестве передаточной функции его чувствительного элемента, полученной после анализа его динамики в выражениях амплитудно-фазовых переменных. Эффективность полученной передаточной функции оптимального регулятора обратной связи продемонстрировано посредством численного моделирования.

Statement of purpose

Coriolis vibratory gyroscopes (CVGs) are interesting due to the possibility to fabricate sensitive elements of such gyroscopes in miniature form by using modern microelectronic mass-production technologies. Such gyroscopes are frequently referred to as MEMS (Micro-Electro-Mechanical-Systems) gyroscopes [1].

CVGs sense Coriolis acceleration, which arises due to the rotation in oscillating structures, it causes a lot more complicated mathematical models, comparing to the conventional types of gyroscopes. One of such complication is a result of the useful signal proportional to the external angular rate being modulated with the intentionally excited primary oscillations. From the control systems point of view, input to dynamic system is primary oscillation excitation signal and unknown angular rate is represented as coefficients of system transfer functions. Due to this conventional control and filtering systems design is practically impossible.

At the same time, performances of CVGs are limited. In view of this problem, optimal controller development is highly necessary. The latter could be achieved only in systems where unknown angular rate is no longer a system parameter but its input [2].

This paper briefly describes method of synthesis of optimal controller design in frequency domain for CVGs sensitive element.

Problem formulation

To synthesize optimal controllers for CVGs the following major steps must be completed:

– development of the mathematical model in demodulated signals;

– obtaining system transfer functions where angular rate is an input;

– synthesis of optimal controllers based on the obtained earlier transfer functions;

– numerical simulations proving the performances of the optimal controller.

Demodulated dynamics of Coriolis vibratory gyroscopes

In the most generalized form, motion equations of the CVG sensitive element both with translational and rotational motion could be represented in the following form [1]:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1 k_1 \dot{x}_1 + (k_1^2 - d_1\Omega^2)x_1 + \\ + g_1\Omega\dot{x}_2 + d_3\dot{\Omega}x_2 = q_1(t), \\ \ddot{x}_2 + 2\zeta_2 k_2 \dot{x}_2 + (k_2^2 - d_2\Omega^2)x_2 - \\ - g_2\Omega\dot{x}_1 - \dot{\Omega}x_1 = q_2(t). \end{cases} \quad (1)$$

where x_1 and x_2 are the generalized coordinates that describe primary (excited) and secondary (sensed) motions of the sensitive element respectively;

ζ_1 and ζ_2 are the dimensionless relative damping coefficients;

k_1 and k_2 are the corresponding natural frequencies;

Ω is the measured angular rate, which is orthogonal to the axes of primary and secondary motions;

q_1 and q_2 are the generalized accelerations due to the external forces acting on the sensitive element.

The remaining dimensionless coefficients are different for the sensitive elements exploiting either translational or rotational motion. For the translational sensitive element they are:

$d_1 = d_2 = 1$;

$d_3 = m_2 / (m_1 + m_2)$;

$g_1 = 2m_2 / (m_1 + m_2)$;

$g_2 = 2$,

where m_1 and m_2 are the masses of the outer frame and the internal massive element.

Using amplitude-phase substitutions for primary and secondary generalized displacements of CVG sensitive element one can obtain from system (1) equation for the secondary oscillation:

$$\ddot{A}_2 + 2(\zeta_2 k_2 + j\omega)\dot{A}_2 + (k_2^2 - \omega^2 + \\ +2j\omega k_2\zeta_2)A_2 = (j\omega g_2\Omega + \dot{\Omega})A_1. \quad (2)$$

Equation (2) describes amplitude-phase of the secondary oscillations with respect to the settled primary oscillations, where A_1 is a constant parameter [2]. This allows to pass on to system transfer functions derivation.

System transfer functions

Having CVG sensitive element motion equation in the form (2), allows obtaining its transfer functions from the input angular rate to the amplitude of the secondary oscillations.

Application of the Laplace transformation to the equation (2) with respect to zero initial conditions for all time-dependent variables results in the following expressions:

$$[(s+jw)^2 + 2k_2 z_2(s+jw) + k_2^2]A_2(s) = \\ = A_1[s+jg_2 w]W(s). \quad (3)$$

Solution of the algebraic equation (3) for the secondary amplitude-phase Laplace transform is

$$A_2(s) = \frac{A_1(s+jg_2 w)}{(s+jw)^2 + 2k_2 z_2(s+jw) + k_2^2} W(s).$$

Considering the angular rate as an input, the system transfer function for the secondary amplitude-phase is

$$W_2(s) = \frac{A_2(s)}{\Omega(s)} = \frac{A_1(s+jg_2\omega)}{(s+j\omega)^2 + 2k_2\zeta_2(s+j\omega) + k_2^2} = \\ = \frac{q_{10}(s+jg_2\omega)}{((s+j\omega)^2 + 2k_2\zeta_2(s+j\omega) + k_2^2)(k_1^2 - \omega^2 + 2j\omega k_1\zeta_1)}. \quad (4)$$

One should note that transfer function (4) has complex coefficients, which results in the complex system outputs as well. There is quite an important special case, when complex transfer function (4) transform to the simple real-valued one. Assuming equal primary and secondary natural frequencies ($k_1 = k_2 = k$), equal damping ratios ($\zeta_1 = \zeta_2 = \zeta$), resonance excitation ($\omega = k$), and constant angular rate, one can easily obtain [2]:

$$W_{20}(s) = \frac{A_{20}(s)}{\Omega(s)} = \frac{q_{10} g_2}{4k^2\zeta(s+k)}. \quad (5)$$

Transfer function (5) relates angular rate to the secondary oscillations amplitude. This transfer function is now useful for derivation transfer function of feedback optimal controller.

Optimal controller transfer function synthesis algorithm

Let's synthesize optimal feedback controller for the CVG analytically. We'll use so called Wiener-Kolmogorov procedure to solve the task of optimal compensation.

CVG motion is possible to describe with the help of the following equation:

$$Px = Mu + M_{\Omega}\Omega \qquad (6)$$

where *x* is output of system;

u is input of system;

Ω is unknown angular rate, which in our case is regarded as disturbance;

P is characteristic polynomial of obtained earlier transfer function (5), angular rate is considered as random process with known spectral density $S_{\Omega\Omega}$.

In our case we consider output *x* of the system as an error, which will be minimized. Measurement in the compensated system is conducted in the feedback contour after passing the signal through the controller.

Assume, that output *x* in fig. 1 is measured by ideal measuring instrument and get into controller, which is situated in feedback and has desired transfer function *W*.

Equation of controller

$$u = Wx. \qquad (7)$$

Let's denote transfer function of closed-loop system from input Щ to output *x* as F_x and transfer function of closed-loop system from input Щ to output *u* as F_u.

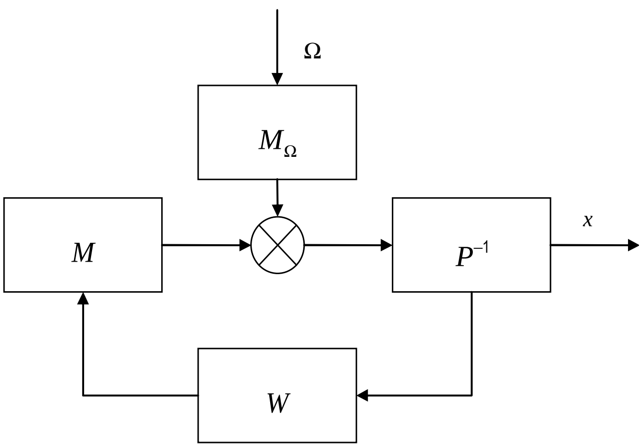

Fig. 1. Structural scheme of "ideal" stabilization system

Substituting u (7) into equation of object (6) we'll obtain

$$F_x = (P - MW)^{-1}; \qquad (8)$$

$$F_u = W(P - MW)^{-1}. \qquad (9)$$

Knowing transfer functions (8) and (9) one can easily obtain equation of feedback controller W.

The task of synthesis is in follows: to define such structure of W on class of fractional-rational functions, which provides stability of closed-loop system and simultaneously deliver minimum to functional:

$$e = \frac{1}{j}\int_{-j\infty}^{j\infty} tr[RS'_{xx} + CS'_{uu}]ds \qquad (10)$$

where S'_{xx}, S'_{uu} are spectral densities of *x* and *u;*

R and *C* are weight positively and negatively defined symmetrical matrices.

As *x* and *u* are equal

$$x = F_x\Omega; \quad u = F_u\Omega,$$

then equation (6) can be rewritten as

$$PF_x - MF_u = E_n \qquad (11)$$

and named as coupling equation between functions F_x and F_u, and functional (10) – as

$$e = \frac{1}{j}\int_{-j\infty}^{j\infty} tr[(F_{x*}RF_x + F_{u*}CF_u)S'_{\Omega\Omega}]\,ds. \qquad (12)$$

Let's form a system of equations

$$\begin{cases} PF_x - MF_u = E_n, \\ PF_x + MF_u = V, \end{cases} \qquad (13)$$

where first equation is coupling equation (11) and second forms function *V*, which varies.

Functions of system F_x and F_u is necessary to express in terms of *V***.** For this purpose let's solve system (13) relatively to matrices F_x and F_u, after that

$$F_x = \frac{1}{2}P^{-1}(V + E_n); \qquad (14)$$

$$F_u = \frac{1}{2}M^{-1}(V - E_n). \qquad (15)$$

After determination of matrices F_x and F_u it will be possible to define transfer function of controller :

$$W = F_u F_x^{-1}. \quad (16)$$

Using equations (14), (15) and (16) we can write structure of transfer function of controller as

$$W = M^{-1}(V - E_n)(V + E_n)^{-1}P.$$

When we'll find structure of V as result of solving of variational task and substitute it into (14), (15) and (16) we can define functions F_x, F_u and W at once.

Substituting functions (14) and (15) into functional (12), we can rewrite it as

$$e = \frac{1}{4j}\int_{-j\infty}^{j\infty} tr\{[(E_n + V_*)P_*^{-1}RP^{-1}(E_n + V) + $$
$$+(V_* - E_n)M_*^{-1}CM^{-1}(V - E_n)]S'_{\Omega\Omega}]\}ds. \quad (17)$$

Let's solve the task with the help of Wiener-Kolmogorov procedure and apply result to our concrete system. The first variation of the functional (17) is [3]:

$$\delta e = \frac{1}{4j}\int_{-j\omega}^{j\omega} tr\{\delta V_*[(P_*^{-1}RP^{-1} + M_*^{-1}CM^{-1})V +$$
$$+(P_*^{-1}RP^{-1} - M_*^{-1}CM^{-1})]S'_{\Omega\Omega} +$$
$$+S'_{\Omega\Omega}[V_*[(P_*^{-1}RP^{-1} + M_*^{-1}CM^{-1}) +$$
$$+(P_*^{-1}RP^{-1} - M_*^{-1}CM^{-1})]\delta V\}ds,$$

where for our case spectral density of angular rate is defined as:

$$S_{\Omega\Omega} = \frac{B^2 g_2^2 q_{10}^2 \sigma^2}{16\zeta^2 k^4 (B^2 - s^2)}. \quad (18)$$

Let's define transposed equation (18) according procedure, which was mentioned above:

$$S'_{\Omega\Omega} = D_* D, \quad (19)$$

where D_* is unstable part,

D is stable part.

Symbol "*" designates Hermite conjugate.

Let's use operation of factorization to devide equation (19) into stable and unstable part.

It'll gives us:

$$D = \frac{Bg_2 q_{10}\sigma}{4\zeta k^2 (B + s)}; \quad (20)$$

$$D_* = \frac{Bg_2 q_{10}\sigma}{4\zeta k^2 (B - s)}.$$

Let's proceed solving of variational task:

$$\Gamma_* \Gamma = P_*^{-1}RP^{-1} + M_*^{-1}CM^{-1}. \quad (21)$$

Factorization of equation (21) gives us:

$$\Gamma = \frac{\sqrt{C}(\sqrt{\zeta^2 k^2 + \frac{M^2}{C}} + s)}{\zeta k + s}, \quad (22)$$

$$\Gamma_* = \frac{\sqrt{C}(\sqrt{\zeta^2 k^2 + \frac{M^2}{C}} - s)}{\zeta k - s}.$$

Another step of Wiener-Kolmogorov procedure:

$$T = T_0 + T_+ + T_- =$$
$$= \Gamma_*^{-1}(P_*^{-1}RP^{-1} - M_*^{-1}CM^{-1})D, \quad (23)$$

where T_0 is number or polynomial;

T_+ is proper fraction with poles in the LHP;

T_- is proper fraction with poles in the RHP.

One can easily obtain T_0, T_+, T_- applying operation of separation to equation (23):

$$T_0 = 0,$$

$$T_+ = \frac{Bg_2 q_{10}\sigma(A_1 s + A_2)}{4\zeta k^2 M\sqrt{C}(\zeta k + s)(B + s)}, \quad (24)$$

where

$$A_1 = \sqrt{C}(-1 +$$
$$+\frac{2M^2}{M^2 + (B + \zeta k)(\zeta kC + \sqrt{C^2\zeta^2 k^2 + M^2 C}}),$$

$$A_2 = -\sqrt{C\zeta^2 k^2 + M^2} +$$
$$+\frac{2M^2(\sqrt{C}(B + \zeta k) + \sqrt{C\zeta^2 k^2 + M^2})}{M^2 + (B + \zeta k)(\zeta kc + \sqrt{C^2\zeta^2 k^2 + M^2 C})}.$$

Now we have all information to define optimal structure of function V as:

$$V = -\Gamma^{-1}(T_0 + T_+)D^{-1}. \quad (25)$$

After substitution equations (20), (22), (24) into equation (25), we'll obtain:

$$V = -\frac{A_1 s + A_2}{(\sqrt{C^2\zeta^2k^2 + CM^2} + s)M}. \quad (26)$$

Structures of transfer functions F_x, F_u and finally optimal feedback controller now can be defined without any trouble substituting equation (26) into equations (14), (15) and (16):

$$F_u = \frac{B + \zeta k + \sqrt{\zeta^2 k^2 + \frac{M^2}{C} + s}}{(M^2 + (B+\zeta k)(\zeta k + \sqrt{C(M^2+\zeta^2k^2C)}))} \times$$
$$\times \frac{-M}{(\sqrt{\zeta^2k^2 + \frac{M^2}{C}} + s)},$$

$$F_x = \frac{\zeta k C + \sqrt{\zeta^2k^2C^2 + M^2C}}{(M^2 + (B+\zeta k)(\zeta k C + \sqrt{C(M^2+\zeta^2k^2C)}))} \times$$
$$\times \frac{B + \zeta k}{(\sqrt{\zeta^2k^2 + \frac{M^2}{C}} + s)},$$

$$W = \frac{-M(B + \zeta k + \sqrt{\zeta^2k^2 + \frac{M^2}{C}} + s)}{(B+\zeta k)(\zeta k C + \sqrt{C(M^2+\zeta^2k^2C)})}. \quad (27)$$

Equation (27) is the final goal of the research, let's now make proving of its useful properties by the numerical simulations.

Numerical simulations

Let's demonstrate efficiency of the obtained optimal feedback controller making numerical simulation.

In order to obtain the most realistic simulation results, equations (1) were used to build a numerical model of CVG dynamics using Simulink/Matlab. Resulting sensitive element model and controller in feedback are shown in the fig. 2 for constant angular rate.

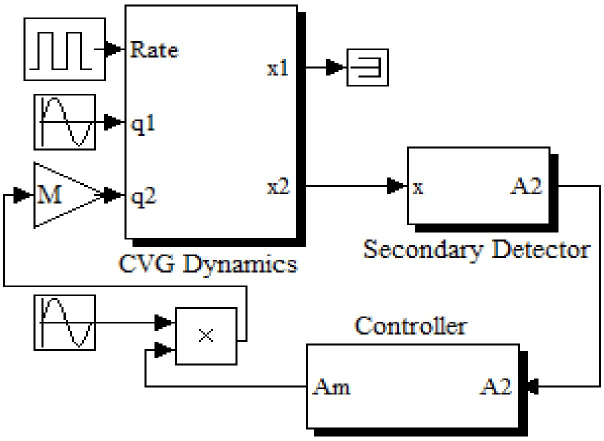

Fig. 2. Simulation model for constant angular rate

Simulation results for model, which is shown in the fig. 2 are demonstrated in the fig. 3.

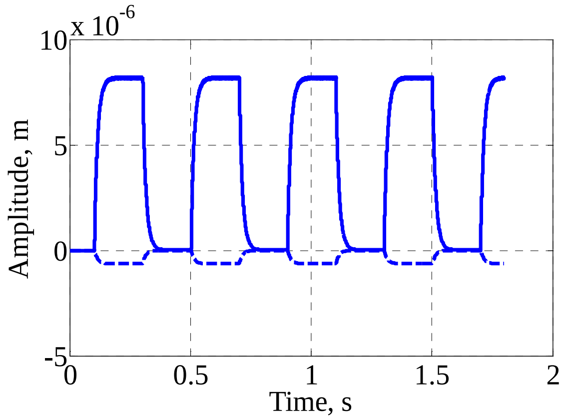

Fig. 3. Simulation results for constant angular rate

Where solid line shows amplitude of the secondary oscillations proportional to the unknown constant angular rate and dashed line shows amplitude of the compensated oscillations.

Analyzing results in the fig. 3, one can see good efficiency of the optimal feedback regulator.

Let's consider at the input of the system not constant, but varying angular rate. Resulting sensitive element model and controller in feedback are shown in the fig. 4 for varying angular rate.

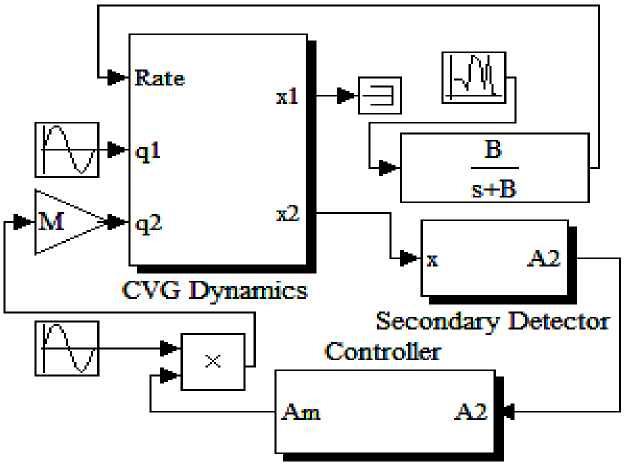

Fig. 4. Simulation model for varying angular rate

Simulation results for model, which is shown in the fig. 4 are demonstrated in the fig. 5.

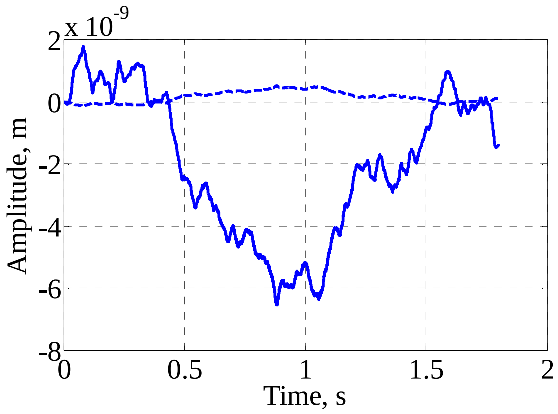

Fig. 5. Simulation results for varying angular rate

Where solid line shows amplitude of the secondary oscillations proportional to the unknown varying angular rate and dashed line shows amplitude of the compensated oscillations.

Analyzing results in the fig. 5, one can see good efficiency of the optimal feedback regulator for input varying angular rate as well.

Conclusion

Presented above synthesis of compensated CVG using Wiener-Kolmogorov procedure resulted in obtaining transfer function of optimal feedback controller. Excellent performance of the regulator has been demonstrated using numerical simulations under action of constant and varying angular rates.

As a future research, further improvement of the controller using slightly modified synthesis algorithm allows obtaining simultaneously features of filter and controller as well is suggested.

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