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TRANSIENT PROCESS ANALYSIS OF CORIOLIS VIBRATORY GYROSCOPES

Introduction

Coriolis vibratory gyroscopes (CVGs) received significant amount of interest from the both scientific and engineering communities due to the possibility to fabricate sensitive elements of such gyroscopes in miniature form by using modern microelectronic mass-production technologies. Such gyroscopes are frequently referred to as MEMS (Micro-Electro-Mechanical-Systems) gyroscopes. Being based on sensing of Coriolis acceleration due to the rotation in oscillating structures, CVGs have a lot more complicated mathematical models, comparing to the conventional types of gyroscopes. One of such complication is a result of the useful signal proportional to the external angular rate being modulated with the intentionally excited primary oscillations [1-3]. From the mathematical modelling point of view, this leads to necessity to "demodulate" the solution in terms of the sensitive element displacements to obtain practically feasible insights into CVG dynamics and errors. From the control systems point of view, conventional representation of CVGs incorporates primary oscillation excitation signal as an input to the dynamic system, and unknown angular rate as a coefficients of its transfer functions [3]. As a result, dynamics of CVGs has been analysed mainly in steady state, while transient process analysis has been omitted due to its apparent complexity.

This paper describes new method of CVG dynamics analysis by means of complex amplitude-phase variables, which enables having angular rate as an input to the dynamic system, and conducting proper transient process analysis without any additional demodulation.

Problem formulation

In the most generalized form, motion equations of the CVG sensitive element both with translational and rotational motion could be represented in the following form [4]:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1 k_1 \dot{x}_1 + (k_1^2 - d_1\Omega^2)x_1 + g_1\Omega\dot{x}_2 + d_3\dot{\Omega}x_2 = q_1(t), \\ \ddot{x}_2 + 2\zeta_2 k_2 \dot{x}_2 + (k_2^2 - d_2\Omega^2)x_2 - g_2\Omega\dot{x}_1 - \dot{\Omega}x_1 = q_2(t). \end{cases} \quad (1)$$

Here x_1 and x_2 are the generalized coordinates that describe primary (excited) and secondary (sensed) motions of the sensitive element respectively, k_1 and k_2

are the corresponding natural frequencies, ζ_1 and ζ_2 are the dimensionless relative damping coefficients, Ω is the measured angular rate, which is orthogonal to the axes of primary and secondary motions, q_1 and q_2 are the generalized accelerations due to the external forces acting on the sensitive element. The remaining dimensionless coefficients are different for the sensitive elements exploiting either translational or rotational motion. For the translational sensitive element they are $d_1 = d_2 = 1$, $d_3 = m_2/(m_1 + m_2)$, $g_1 = 2m_2/(m_1 + m_2)$, $g_2 = 2$, where were m_1 and m_2 are the masses of the outer frame and the internal massive element. In case of the rotational motion of the sensitive element, these coefficients are the functions of different moments of inertia (for greater details see [4]).

In the presented above motion equations, the angular rate is included as an unknown and variable coefficient rather than an input to the double oscillator system. In order to identify the angular rate one must detect secondary oscillations of the sensitive element and measure its amplitude, which is approximately directly proportional to the angular rate, and phase, which gives the sign.

In order to make the equations (1) suitable for to the transient process analysis we must make the following assumptions: angular rate is small comparing to the primary and secondary natural frequencies so that

$$k_1^2 >> d_1\Omega^2,\ k_2^2 >> d_2\Omega^2 \tag{2}$$

and rotational and Coriolis accelerations acting along primary oscillation axis are negligible in comparison to the accelerations from driving forces

$$g_1\Omega\dot{x}_2 + d_3\dot{\Omega}x_2 << q_1(t). \tag{3}$$

Taking into considerations assumptions (2) and (3), motions equations (1) could be simplified to the following form:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1 k_1 \dot{x}_1 + k_1^2 x_1 = q_1(t), \\ \ddot{x}_2 + 2\zeta_2 k_2 \dot{x}_2 + k_2^2 x_2 = g_2\Omega\dot{x}_1 + \dot{\Omega}x_1. \end{cases} \tag{4}$$

Here we also assumed that no external driving forces are affecting the secondary oscillations, which means that $q_2(t) = 0$. System of equations (4) is now perfectly suitable for further transformations towards the desired representation in terms of the unknown angular rate.

Amplitude-phase motion equations

As has been shown in [5], by means of a proper chosen phase shift of the excitation voltage applied to the sensitive element, the excitation force could be shaped to the perfect harmonic form. Using exponential representation of complex numbers, such a driving force $q_1(t)$ could be represented as

$$q_1(t) = q_{10}\sin(\omega t) = \mathrm{Im}\{q_{10}e^{j\omega t}\}. \tag{5}$$

Here ω is the excitation frequency given in radians per second, q_{10} is the constant excitation acceleration amplitude. Non-homogeneous solutions of the motion equations (1) or (4) for primary and secondary oscillations are searched in a similar form

$$\begin{aligned} x_1(t) &= \operatorname{Im}\{A_1(t)e^{j\omega t}\}, \quad A_1(t) = A_{10}(t)e^{j\varphi_{10}(t)}, \\ x_2(t) &= \operatorname{Im}\{A_2(t)e^{j\omega t}\}, \quad A_2(t) = A_{20}(t)e^{j\varphi_{20}(t)}, \end{aligned} \tag{6}$$

where A_{10} and A_{20} are the primary and secondary oscillation amplitudes, φ_{10} and φ_{20} are the corresponding phase shifts relatively to the excitation force. Although these quantities are real (non-imaginary), they are combined in complex amplitude-phase variables A_1 and A_2.

Substituting expressions (5) and (6) into equations (4) results in the following motions equations in terms of the complex amplitude-phase variables rather than real generalized coordinates:

$$\begin{cases} \ddot{A}_1 + 2(\zeta_1 k_1 + j\omega)\dot{A}_1 + (k_1^2 - \omega^2 + 2j\omega k_1\zeta_1)A_1 = q_{10}, \\ \ddot{A}_2 + 2(\zeta_2 k_2 + j\omega)\dot{A}_2 + (k_2^2 - \omega^2 + 2j\omega k_2\zeta_2)A_2 = (j\omega g_2\Omega + \dot{\Omega})A_1 + g_2\dot{A}_1\Omega. \end{cases} \tag{7}$$

Equations (7) describe variations of the amplitude and phase of the primary and secondary equations in time with respect to the unknown non-constant angular rate $\Omega(t)$. This allows conducting analysis of the Coriolis vibratory gyroscope dynamics without constraining the angular rate to be constant or slowly varying.

Analysing system (7), one can see that the first equation can be solved separately from the second one. After homogeneous solutions of the first equation faded out, only non-homogenous solution remains. In this case, amplitude of the primary oscillations is

$$A_1 = \frac{q_{10}}{k_1^2 - \omega^2 + 2jk_1\zeta_1\omega}, \tag{8}$$

and it is constant in time, yielding $\ddot{A}_1 = \dot{A}_1 = 0$. Indeed, most of the time measurements of the angular rate are performed when primary oscillations have already settled. As a result, only equation for the secondary oscillations remains, in which the complex primary amplitude A_1 is just a constant parameter given by (8):

$$\ddot{A}_2 + 2(\zeta_2 k_2 + j\omega)\dot{A}_2 + (k_2^2 - \omega^2 + 2j\omega k_2\zeta_2)A_2 = (j\omega g_2\Omega + \dot{\Omega})A_1. \tag{9}$$

Equation (7) now describes amplitude-phase of the secondary oscillations with respect to the settled primary oscillations.

System transfer functions

Having CVG sensitive element motion equation in the form (9), allows analysis of its transient processes in amplitudes and phases with respect to arbitrary angular rates applied to the system. Application of the Laplace transformation to the equations (7) with respect to zero initial conditions for all time-dependent variables results in the following expressions:

$$[(s+j\omega)^2+2\zeta_2 k_2(s+j\omega)+k_2^2]A_2(s)=A_1[s+jg_2\omega]\Omega(s). \tag{10}$$

Solution of the algebraic equation (10) for the secondary amplitude-phase Laplace transform is

$$A_2(s)=\frac{A_1(s+jg_2\omega)}{(s+j\omega)^2+2\zeta_2 k_2(s+j\omega)+k_2^2}\Omega(s). \tag{11}$$

Considering the angular rate as an input, the system transfer function for the secondary amplitude-phase is

$$\begin{aligned}W_2(s)&=\frac{A_2(s)}{\Omega(s)}=\frac{A_1(s+jg_2\omega)}{(s+j\omega)^2+2\zeta_2 k_2(s+j\omega)+k_2^2}=\\&=\frac{q_{10}(s+jg_2\omega)}{[(s+j\omega)^2+2\zeta_2 k_2(s+j\omega)+k_2^2][k_1^2-\omega^2+2j\omega k_1\zeta_1]}.\end{aligned} \tag{12}$$

One should note that transfer function (12) has complex coefficients, which results in the complex system outputs as well. Although it is somewhat unusual, it still enables us to analyse CVG dynamics and transient processes due to the angular rate in an open-loop dynamic system.

Amplitude and phase responses

In order to calculate the amplitude response of the system using transfer function (12), Laplace variable s must be replaced with the Fourier variable $j\lambda$, where λ is the frequency of the angular rate oscillations:

$$W_2(j\lambda)=\frac{jq_{10}(\lambda+g_2\omega)}{[k_2^2-(\lambda+\omega)^2+2j\zeta_2 k_2(\lambda+\omega)][k_1^2-\omega^2+2j\omega\zeta_1 k_1]}. \tag{13}$$

Absolute value of the complex function (13) is the amplitude response of the secondary oscillations amplitude to the harmonic angular rate, and the corresponding phase of the complex function is the phase response:

$$\begin{aligned}A(\lambda)&=\frac{q_{10}(\lambda+g_2\omega)}{\sqrt{[(k_2^2-(\lambda+\omega)^2)^2+4\zeta_2^2k_2^2(\lambda+\omega)^2][(k_1^2-\omega^2)^2+4\zeta_1^2k_1^2\omega^2]}}\\ \varphi(\lambda)&=\tan^{-1}\left\{\frac{[k_2^2-(\lambda+\omega)^2][k_1^2-\omega^2]-4k_1k_2\zeta_1\zeta_2\omega(\lambda+\omega)}{2[k_2\zeta_2(\lambda+\omega)(k_1^2-\omega^2)+k_1\zeta_1\omega(k_2^2-(\lambda+\omega)^2)]}\right\}.\end{aligned} \tag{14}$$

One should note that, assuming constant angular rate ($\lambda=0$) in the expressions (12) the well known expressions ([5]) for the amplitude and phase of the secondary oscillations is obtained.

Analysis of the expressions (12) shows that effect from the oscillating angular rate is practically equivalent to shift of the excitation frequency by the frequency of the angular rate. This causes CVGs, especially those with high Q-factor, to loose its resonant tuning, which in turn results in significant variation of its scale factor (dynamic error). Solution of this problem by means of proper choice of natural frequency split and damping has been suggested in [6].

Stability and transient process optimisation

Both stability and unit-step transient process quality depend on position of the system poles in the real-imaginary plane. Poles of the transfer function (12) are as follows:

$$s_{1,2} = -k_2\zeta_2 \pm jk_2\sqrt{1-\zeta_2^2} - j\omega, \tag{15}$$

Analysing expressions (15), it is easy to see that CVGs are inherently stable. Indeed, if the relative damping coefficient $\zeta_2 \le 1$, then real parts of the poles are $-k_2\zeta_2 < 0$. If the relative damping coefficient $\zeta_2 > 1$, then real parts are $-k_2(\zeta_2 \pm \sqrt{\zeta_1^2 - 1}) < 0$.

Ideal (half-oscillatory) unit-step angular rate transient process in secondary oscillations amplitude is achievable if imaginary parts of the poles (15) are zero. One pole has large imaginary part $-k_2\sqrt{1-\zeta_2^2} - \omega < 0$, which is always way below zero, and corresponds to high frequency oscillations in the envelope. The second pole is responsible for the low frequency oscillations, and is the most essential for the transient process. For this pole the ideal transient process condition has the following form:

$$k_2\sqrt{1-\zeta_2^2} - \omega = 0 \Rightarrow k_2 = \frac{\omega}{\sqrt{1-\zeta_2^2}}.$$

For example, if primary oscillations are excited in pure resonance for better sensitivity (see [5]), equation is transformed to

$$k_2 = k_1\sqrt{\frac{1-2\zeta_1^2}{1-\zeta_2^2}}. \tag{16}$$

As a result, in order to provide ideal transient process for the secondary oscillations amplitude, one should design sensitive element of CVG with the natural frequency of the secondary oscillations according to (16).

Another important performance feature of a system transient process is its settling time, which is defined by the real part of the system poles and can be approximated as

$$T = -\frac{\ln(\varepsilon)}{k_2\zeta_2}.$$

Here ε is the error tolerance (ε=0.01 for 1% tolerance). From this dependence one can see, that in order to minimize settling time, denominator $k_2\zeta_2$ must be maximised. Since sensitivity of CVG is inversely related to its natural frequencies (see [5]), reducing its damping along with the natural frequencies will increase its transient process settling time.

Case of slowly varying amplitude

Another consequence of the presented above analysis of the system poles and its transient process is that actual amplitude of the secondary oscillations is mainly defined by the low frequency pole, while effect from the high frequency pole can be neglected, since it will be removed during demodulation process. In other words, predominant behaviour is a slow variation of the amplitude and phase. Neglecting the second order derivative in the equation (9) yields

$$2(\zeta_2 k_2 + j\omega)\dot{A}_2 + (k_2^2 - \omega^2 + 2j\omega k_2\zeta_2)A_2 = (j\omega g_2\Omega + \dot{\Omega})A_1, \quad (17)$$

and the corresponding angular rate transfer function becomes

$$W_2(s) = \frac{q_{10}(s + jg_2\omega)}{[2\zeta_2 k_2 s + k_2^2 - \omega^2 + j2\omega(\zeta_2 k_2 + s)][k_1^2 - \omega^2 + 2j\omega k_1\zeta_1]}. \quad (18)$$

Single pole of this transfer function is

$$s_1 = -\frac{k_2^2 - \omega^2 + 2j\omega\zeta_2 k_2}{2(\zeta_2 k_2 + j\omega)} = -k_2\zeta_2\frac{k_2^2 + \omega^2}{2(\zeta_2^2 k_2^2 + \omega^2)} + j\frac{k_2^2\omega - 2k_2^2\zeta_2^2\omega - \omega^3}{2(\zeta_2^2 k_2^2 + \omega^2)}. \quad (19)$$

Ideal unit-step transient process achieved when imaginary part of the (19) equals to zero, which in turn gives

$$\omega = k_2\sqrt{1 - 2\zeta_2^2}, \quad (20)$$

which is the eigenfrequency of the secondary oscillations. However, as it has been mentioned earlier, better sensitivity is achieved when the sensitive element is driven in the primary resonance, which means $\omega = k_1\sqrt{1 - 2\zeta_1^2}$. In this case

$$k_2 = k_1\sqrt{\frac{1 - 2\zeta_1^2}{1 - 2\zeta_2^2}}. \quad (21)$$

Although this formula is somewhat different from the obtained earlier dependence (16), actual values are quite close. If the secondary natural frequency is chosen accordingly to (21) then pole (19) becomes

$$s_1 = -k_1\zeta_2\sqrt{\frac{1 - 2\zeta_1^2}{1 - 2\zeta_2^2}} = -k_2\zeta_2. \quad (22)$$

Obviously, unit-step settling time is still given by the expression (16), and all the hints to the settling time minimization remain the same.

Real and imaginary transfer functions

While simulating dynamics of CVG based on the complex amplitude-phase transfer functions (12) or (18) one could have problems dealing with complex coefficients of these transfer functions. One way to avoid this problem is to consider real and imaginary parts of complex amplitude as separate signals, which are then combined together to produce real amplitude and phase. In order to obtain transfer functions for such signals let us represent primary and secondary amplitudes as:

$$A_1 = A_{1R} + jA_{1I}, \; A_2 = A_{2R} + jA_{2I}. \quad (23)$$

Primary oscillations components can be easily found by means of substituting expressions (23) into (8)

$$A_{1R} = \frac{q_{10}(k_1^2 - \omega^2)}{(k_1^2 - \omega^2)^2 + 4k_1^2\zeta_1^2\omega^2}, \quad A_{1I} = -\frac{2q_{10}j\omega k_1\zeta_1}{(k_1^2 - \omega^2)^2 + 4k_1^2\zeta_1^2\omega^2}. \quad (24)$$

At the same time, substituting expressions (23) into the motion equation (9), and applying Laplace transformation gives

$$\begin{cases}(k_2^2-\omega^2+2k_2\zeta_2 s+s^2)A_{2R}-2\omega(k_2\zeta_2+s)A_{2I}=(A_{1R}s-A_{1I}g_2\omega)\Omega,\\(k_2^2-\omega^2+2k_2\zeta_2 s+s^2)A_{2I}+2\omega(k_2\zeta_2+s)A_{2R}=(A_{1I}s+A_{1R}g_2\omega)\Omega.\end{cases} \tag{25}$$

Resolving algebraic system (25) with respect to unknown real and imaginary parts of the secondary complex amplitude yields

$$A_{2R}=\frac{A_{1R}M_{RR}(s)+A_{1I}M_{RI}(s)}{P(s)}\Omega,\quad A_{2I}=\frac{A_{1R}M_{IR}(s)+A_{1I}M_{II}(s)}{P(s)}\Omega. \tag{26}$$

Here the numerator polynomials from the real and imaginary parts of primary amplitudes are given by the following expressions:

$$\begin{aligned}M_{RR}(s)&=s(k_2^2+2k_2\zeta_2 s+s^2)-\omega^2(s-2g_2(s+k_2\zeta_2))\\M_{RI}(s)&=\omega[2s(s+k_2\zeta_2)-g_2(k_2^2-\omega^2+2k_2\zeta_2 s+s^2)],\\M_{II}(s)&=2\omega^2 g_2(s+k_2\zeta_2)+s(k_2^2-\omega^2+2k_2\zeta_2 s+s^2)],\\M_{IR}(s)&=\omega[g_2(k_2^2-\omega^2+2k_2\zeta_2 s+s^2)-2s(s+k_2\zeta_2)].\\P(s)&=4(s+k_2\zeta_2)^2\omega^2+(k_2^2-\omega^2+2k_2\zeta_2 s+s^2)^2\end{aligned} \tag{27}$$

One should note that if primary oscillations are excited at the primary natural frequency ($\omega=k_1$), then

$$A_{1I}=-\frac{q_{10}}{2k_1^2\zeta_1},\quad A_{1R}=0, \tag{28}$$

and two out of the four transfer functions (27) become unnecessary.

Finally, there is quite an important case, when complex transfer functions transform to the simple real one. Assuming equal primary and secondary natural frequencies ($k_1=k_2=k$), damping ratios ($\zeta_1=\zeta_2=\zeta$), resonance excitation, and constant angular rate, one can easily obtain

$$A_{20}(s)=\sqrt{A_{2R}^2+A_{2I}^2}=\frac{q_{10}g_2}{4k^2\zeta(s+k\zeta)}. \tag{29}$$

Although this case appears to be very specific, it still approximates transient process of a "tuned" CVG with accuracy suitable for most of the applications.

Numerical simulations

In order to obtain the most realistic transient process, a non-linear model of CVG has been implemented based on the equations (1) with neglected centrifugical accelerations according to (2) and added synchronous demodulator (see figure 1). This model is used to verify output produced by the complex model based on the real and imaginary transfer functions (26) and (27) (shown in the figure 2).

Simulation results for these to models are shown in the figures 3 and 4, where solid line corresponds to the complex model output, dotted line corresponds to the "realistic" reference output, and dashed line shows the input angular rate.

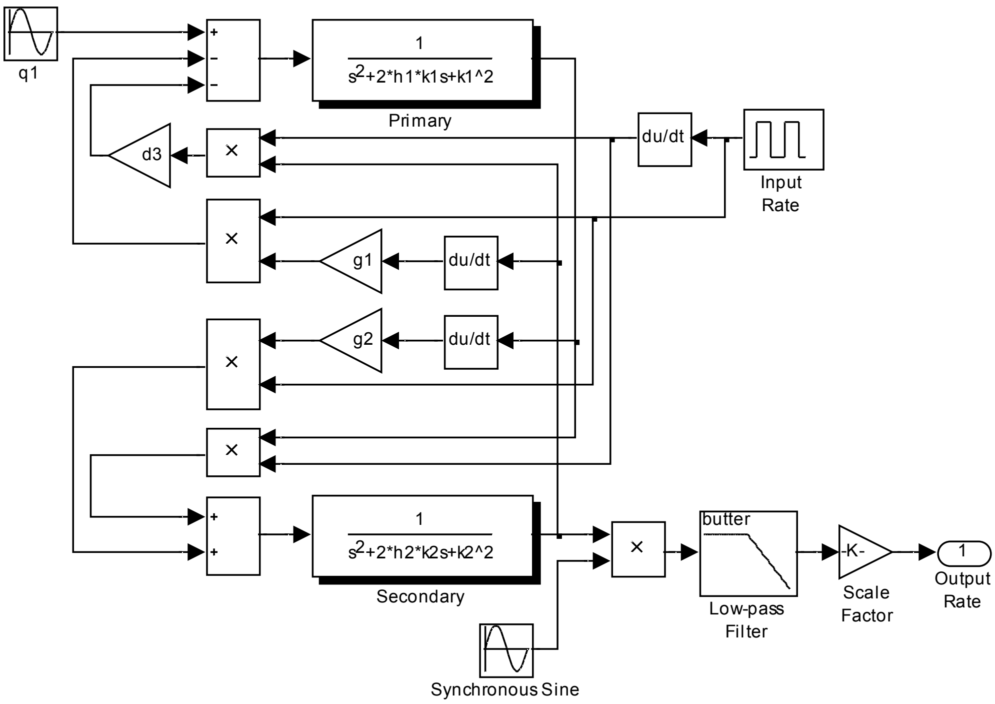

Fig. 1. Realistic CVG simulation model

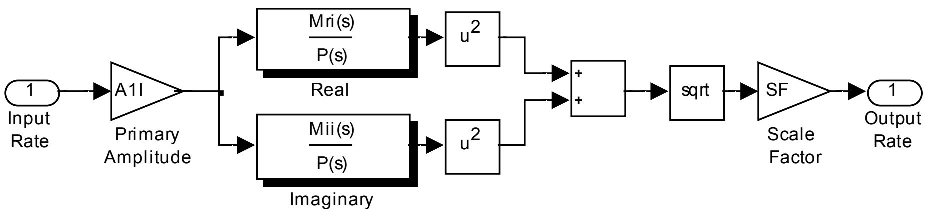

Fig. 2. Real and imaginary transfer functions model

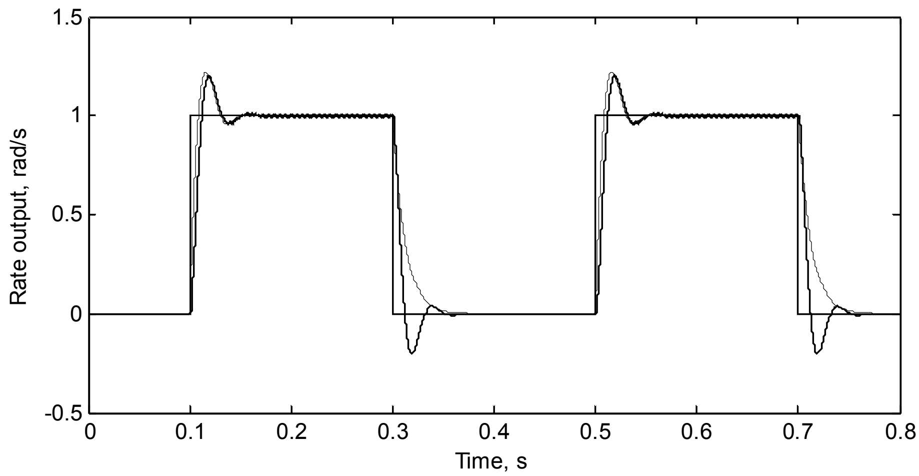

Fig. 3. Non-optimised transient process
($k_1 = 1000\pi$, $k_2 = 1.05k_1$, $\zeta_1 = \zeta_2 = 0.025$, $\omega = k_1$)

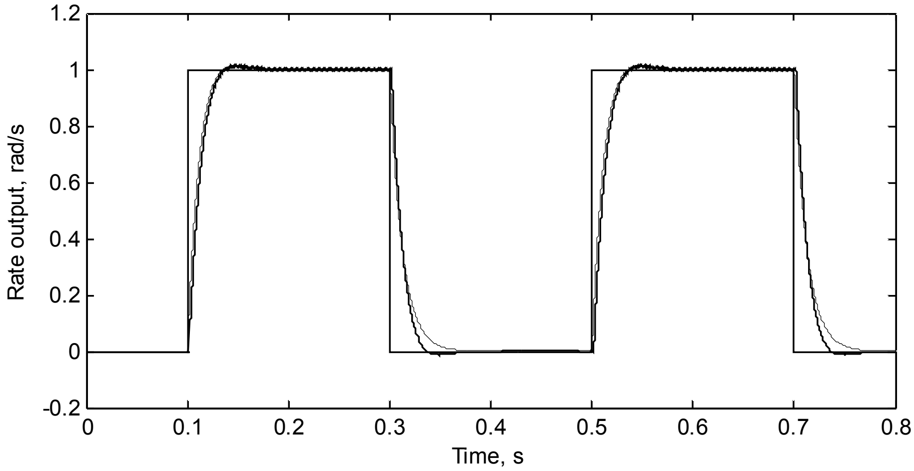

Fig. 4. Optimised transient process
($k_1 = 1000\pi$, $k_2 = 0.987k_1$, $\zeta_1 = \zeta_2 = 0.025$, $\omega = k_1$)

In the first case, transient process for the non-optimised CVG is shown, expressing significant overshoot and clear oscillatory behaviour. In the second case secondary natural frequency is chosen according to (16), which resulted in half-oscillation transient process as expected.

From the graphs in the figures 3 and 4, one can see that “realistic” transient process is somewhat different from the “complex” one. This is believed to be the result of demodulating with the fixed phase shifted signal, while the actual phase shift varies in time. At the same time, “complex” output is much closer to the real secondary oscillations envelope, than the demodulated “realistic”.

Conclusions

Presented above analysis of CVG dynamics using amplitude-phase complex variables resulted in obtaining system transfer functions, where measured angular rate became an input rather than a parameter. This made possible to analyse amplitude and phase responses of CVG and its transient processes in already demodulated signals. As a result, conditions of the optimal transient process have been obtained along with formula for its performances calculation. Excellent performances of the obtained transfer functions have been demonstrated using numerical simulations.

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TRANSIENT PROCESS ANALYSIS OF CORIOLIS VIBRATORY GYROSCOPES

Analysis of the Coriolis vibratory gyroscopes sensitive element dynamics in terms of the amplitude-phase variables leading to the proper transfer functions of such inertial sensors is proposed in this paper. Obtained transfer functions are used to derive expressions for the amplitude and phase response of such sensors, and to conduct analysis and optimisation of their measurement performances in terms of transient processes.
Keywords: Coriolis vibratory gyroscopes, transient process

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АНАЛІЗ ПЕРЕХІДНИХ ПРОЦЕСІВ У КОРІОЛІСОВИХ ВІБРАЦІЙНИХ ГІРОСКОПІВ

В цій статті представлено аналіз динаміки чутливого елемента Коріолісових вібраційних гіроскопів в амплітудно-фазових змінних, що веде до виведення корректних передатних функцій таких інерціальних датчиків. Отримана передатна функція використана для виведення виразів для амплітудно- та фазо-частотних характеристик датчиків, та оптимізації їх вимірювальних характеристик на основі аналізу перехідних процесів.
Ключові слова: коріолісовий вібраційний гіроскоп, перехідний процес

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АНАЛИЗ ПЕРЕХОДНЫХ ПРОЦЕССОВ У КОРИОЛИСОВЫХ ВИБРАЦИОННЫХ ГИРОСКОПОВ

В этой статье представлен анализ динамики чувствительного элемента Кориолисовых вибрационных гироскопов в амплитудно-фазовых переменных, который позволяет получить корректную передаточную функцию таких инерциальных датчиков. Полученная передаточная функция использована для получения выражений для амплитудно- и фазо-частотных характеристик датчиков, а также для оптимизации их измерительных характеристик на основе анализа переходных процессов.
Ключевые слова: кориолисовый вибрационный гироскоп, переходной процесс