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DYNAMIC ERRORS OF CORIOLIS VIBRATORY GYROSCOPES

Introduction

Generalized dynamics of Coriolis vibratory gyroscopes (CVGs) has been under intensive study for last two decades. Such an increased interest has been partially caused by the possibility to fabricate sensitive elements for such gyroscopes using microelectronic mass-production technologies. As a result, new family of MEMS inertial sensors appeared.

Mathematical modelling of the sensitive element took considerable part among other directions of CVG research. Analysis of the sensitive element dynamics in terms of the inertial element displacement is quite complicated [1]. Some useful results were obtained by applying method of averaging to the single-mass system with two degrees of freedom [2]. Later dynamics of the translational sensitive element has been analysed in more generalized form for designs involving additional decoupling frame [3, 4]. Finally combined general model for CVGs both with translational and rotational motion of the sensitive element has been devised in [5], including analytical design methodology for the sensitive element. At the same time, generalized analysis of the CVG as an element of control system although been attempted in [4] yet was inefficient due to the fact that unknown angular rate is a coefficient in the motion equations rather than input to the oscillator.

Approximate dynamic error analysis with application to the bandwidth realization in terms of the generalized CVG model has been suggested in [5]. Main goal of this paper is to present new methodology for the CVG dynamic errors analysis and recommendation for the sensor bandwidth optimization.

Problem formulation

In the most generalized form, motion equations of the CVG sensitive element both with translational and rotational motion could be represented in the following form:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1 k_1 \dot{x}_1 + (k_1^2 - d_1\Omega^2)x_1 + g_1\Omega\dot{x}_2 + d_3\dot{\Omega}x_2 = q_1(t), \\ \ddot{x}_2 + 2\zeta_2 k_2 \dot{x}_2 + (k_2^2 - d_2\Omega^2)x_2 - g_2\Omega\dot{x}_1 - \dot{\Omega}x_1 = q_2(t). \end{cases} \quad (1)$$

Here x_1 and x_2 are the generalized coordinates that describe primary (excited) and secondary (sensed) motions of the sensitive element respectively [5], k_1 and k_2 are the corresponding natural frequencies, ζ_1 and ζ_2 are the dimensionless relative damping coefficients, Ω is the measured angular rate, which is orthogonal to the axes of primary and secondary motions, q_1 and q_2 are the generalized accelerations due to the external forces acting on the sensitive element. The remaining dimensionless coefficients are different for the sensitive elements exploiting either translational or rotational motion. They can be calculated using expressions given in the table 1, were m_1 and m_2 are the masses of the outer frame and internal massive element in case of translational motion, and corresponding moments of inertial for the rotational motion (for greater details see [5]).

Table 1. Dimensionless parameters of the CVG dynamics

	Translational	Rotational
d_1	1	$(I_{12}+I_{23}-I_{12}-I_{22})/(I_{11}+I_{21})$
d_2	1	$(I_{23}-I_{21})/I_{22}$
d_3	$m_2/(m_1+m_2)$	$(I_{21}-I_{23})/(I_{11}+I_{21})$
g_1	$2m_2/(m_1+m_2)$	$(I_{22}+I_{21}-I_{23})/(I_{11}+I_{21})$
g_2	2	$(I_{22}+I_{21}-I_{23})/I_{22}$

In the presented above motion equations the angular rate is included as an unknown and variable coefficient rather than an input to the double oscillator system. In order to identify the angular rate one must detect secondary oscillations of the sensitive element and measure its amplitude, which is approximately directly proportional to the angular rate, and phase, which gives the sign.

In order to make the equations (1) suitable for to the dynamic error analysis we must make the following assumptions: angular rate is small comparing to the primary and secondary natural frequencies so that

$$k_1^2 >> d_1\Omega^2,\ k_2^2 >> d_2\Omega^2 \tag{2}$$

and rotational and Coriolis accelerations are negligible comparing to the accelerations from driving forces

$$g_1\Omega\dot{x}_2 + d_3\dot{\Omega}x_2 << q_1(t). \tag{3}$$

Taking into considerations assumptions (2) and (3), motions equations (1) could be simplified as follows:

$$\begin{cases} \ddot{x}_1 + 2\zeta_1 k_1\dot{x}_1 + k_1^2 x_1 = q_1(t), \\ \ddot{x}_2 + 2\zeta_2 k_2\dot{x}_2 + k_2^2 x_2 = g_2\Omega\dot{x}_1 + \dot{\Omega}x_1. \end{cases} \tag{4}$$

Here we also assumed that no external driving forces are affecting the secondary oscillations, which means that $q_2(t)=0$. System of equations (4) is now perfectly suitable for further transformations towards the desired representation in terms of the unknown angular rate.

Motion equations in amplitude-phase variables

As has been shown in [6], by means of a proper chosen phase shift of the excitation voltage applied to the sensitive element, the excitation force could be shaped to the perfect harmonic form. Using exponential representation of complex numbers, such a driving force $q_1(t)$ could be represented as

$$q_1(t) = q_{10}\sin(\omega t) = \mathrm{Im}\{q_{10}e^{j\omega t}\}. \tag{5}$$

Here ω is the excitation frequency given in radians per second, q_{10} is the constant excitation acceleration amplitude. Non-homogeneous solutions of the motion equations (1) or (4) for primary and secondary oscillations are searched in a similar form

$$\begin{aligned} x_1(t) &= \mathrm{Im}\{A_1(t)e^{j\omega t}\}, \quad A_1(t) = A_{10}(t)e^{j\varphi_{10}(t)}, \\ x_2(t) &= \mathrm{Im}\{A_2(t)e^{j\omega t}\}, \quad A_2(t) = A_{20}(t)e^{j\varphi_{20}(t)}, \end{aligned} \tag{6}$$

where A_{10} and A_{20} are the primary and secondary oscillation amplitudes, φ_{10} and φ_{20} are the corresponding phase shifts relatively to the excitation force. Although these quantities are real, they are combined in complex amplitude-phase variables A_1 and A_2.

Substituting expressions (5) and (6) into equations (4) results in the following motions equations in terms of the complex amplitude-phase variables rather than real generalized coordinates:

$$\begin{cases} \ddot{A}_1 + 2(\zeta_1 k_1 + j\omega)\dot{A}_1 + (k_1^2 - \omega^2 + 2jk_1\zeta_1)A_1 = q_{10}, \\ \ddot{A}_2 + 2(\zeta_2 k_2 + j\omega)\dot{A}_2 + (k_2^2 - \omega^2 + 2jk_2\zeta_2)A_2 = (g_2\Omega j\omega + \dot{\Omega})A_1. \end{cases} \tag{7}$$

Equations (7) describe variations of the amplitude and phase of the primary and secondary equations in time with respect to the unknown non-constant angular rate $\Omega(t)$. This allows conducting analysis of the Coriolis vibratory gyroscope dynamics without constraining the angular rate to be constant or slowly varying.

System transfer functions

Having CVG sensitive element motion equations in the form (7) allows analysis of its transient processes in amplitudes and phases with respect to arbitrary angular rates affecting the system. However, in order to analyse the system dynamic error we need its amplitude response from the angular rate. In order to obtain the amplitude response the system transfer functions must be obtained. Application of the Laplace transformation to the equations (7) with respect to zero initial conditions for all time-dependent variables results in the following expressions:

$$\begin{cases} [(s + j\omega)^2 + 2\zeta_1 k_1(s + j\omega) + k_1^2]A_1(s) = q_{10}, \\ [(s + j\omega)^2 + 2\zeta_2 k_2(s + j\omega) + k_2^2]A_2(s) = A_1(s)[s + jg_2\omega]\Omega(s). \end{cases} \tag{8}$$

Solutions of the algebraic system (8) for the primary and secondary amplitude-phase Laplace transforms are given as

$$A_1(s) = \frac{q_{10}}{(s+j\omega)^2 + 2\zeta_1 k_1 (s+j\omega) + k_1^2},$$
$$A_2(s) = \frac{q_{10}[s + jg_2\omega]\Omega(s)}{[(s+j\omega)^2 + 2\zeta_2 k_2 (s+j\omega) + k_2^2][(s+j\omega)^2 + 2\zeta_1 k_1 (s+j\omega) + k_1^2]}. \quad (9)$$

Considering the angular rate as an input, the system transfer function for the secondary amplitude-phase is

$$W_2(s) = \frac{q_{10}(s + jg_2\omega)}{[(s+j\omega)^2 + 2\zeta_2 k_2 (s+j\omega) + k_2^2][(s+j\omega)^2 + 2\zeta_1 k_1 (s+j\omega) + k_1^2]}. \quad (10)$$

One should note that the transfer function (10) has complex coefficients that results in the complex system output as well. Although it is somewhat unusual, it still enables us to calculate amplitude response of the CVG due to the harmonic angular rate. Apart from that, transfer function itself allows further study of the Coriolis vibratory gyroscopes as an open-loop dynamic system.

Amplitude and phase responses

In order to calculate the amplitude response of the system using transfer function (10), Laplace variable s must be replaced with the Fourier variable $j\lambda$, where λ is the frequency of the angular rate oscillations:

$$W_2(j\lambda) = jq_{10}(\lambda + kg_2\delta\omega)/[k^2\delta k^2 - (\lambda + k\delta\omega)^2 + 2jk\delta k\zeta\delta\zeta(\lambda + k\delta\omega)]\times$$
$$\times[k^2 - (\lambda + k\delta\omega)^2 + 2jk\zeta(\lambda + k\delta\omega)]. \quad (11)$$

Here the new variables are given by the following expressions:

$$k = k_1,\ \delta k = k_2 / k_1,\ \delta\omega = \omega / k_1,\ \zeta = \zeta_1,\ \delta\zeta = \zeta_2 / \zeta_1.$$

Similarly, by introducing the dimensionless relative frequency of the angular rate as $\delta\lambda = \lambda / k_1$, expression (11) can be simplified:

$$W_2(j\delta\lambda) = jq_{10}(\delta\lambda + g_2\delta\omega)/k^3[\delta k^2 - (\delta\lambda + \delta\omega)^2 + 2j\delta k\zeta\delta\zeta(\delta\lambda + \delta\omega)]\times$$
$$\times[1 - (\delta\lambda + \delta\omega)^2 + 2j\zeta(\delta\lambda + \delta\omega)]. \quad (12)$$

Absolute value of the complex function (12) is the amplitude response of the secondary oscillations amplitude to the harmonic angular rate, and the corresponding phase of the complex function is the phase response:

$$A(\delta\lambda) = q_{10}(\delta\lambda + g_2\delta\omega)/k^3[(\delta k^2 - (\delta\lambda + \delta\omega)^2)^2 + 4\delta k^2\zeta^2\delta\zeta^2(\delta\lambda + \delta\omega)^2]^{1/2}\times$$
$$\times[(1 - (\delta\lambda + \delta\omega)^2)^2 + 4\zeta^2(\delta\lambda + \delta\omega)^2]^{1/2}$$

$$\tan\{\varphi(\delta\lambda)\} = \{[\delta k^2 - (\delta\lambda + \delta\omega)^2][1 - (\delta\lambda + \delta\omega)^2] - 4\delta k\zeta^2\delta\zeta(\delta\lambda + \delta\omega)^2\}/$$
$$2\zeta[\delta k\delta\zeta(\delta\lambda + \delta\omega)(1 - (\delta\lambda + \delta\omega)^2) + (\delta k^2 - (\delta\lambda + \delta\omega)^2)(\delta\lambda + \delta\omega)]. \quad (13)$$

One should note that, assuming constant angular rate ($\delta\lambda = 0$) in the expressions (13) the well known expressions ([5]) for the amplitude and phase of the secondary oscillations could be obtained.

Dynamic errors analysis

Dynamic error of the CVG can by analysed in terms of the amplitude distortions due to the angular rate frequency as well as in terms of the corresponding distortions of the phase. In an ideal case, amplitude and phase of the secondary oscillations for the harmonic angular rate must by the same as for the constant one. This allows defining the dynamic error both for the amplitude and phasing as follows:

$$E_A = \frac{A(\delta\lambda)}{A(0)},\; E_\varphi = \frac{\varphi(\delta\lambda)}{\varphi(0)}. \tag{14}$$

Errors (14) are dimensionless and are equal to 1 in the ideal case.

Let us first study the phase dynamic error. Except of the relative angular rate frequency, phase dynamic error depends on such design parameters of the sensitive element as relative excitation frequency $\delta\omega$, natural frequency ratio δk, relative damping ratio $\delta\zeta$, and damping factor of the primary oscillations ζ. As has been shown in [1], it is advantageous to excite the sensitive element at the natural frequency of primary oscillations ($\delta\omega = 1$). In this case graphic plot of the phase dynamic error for different primary damping factors is shown in the figure 1.

Analysis of this graph suggests that although it appears that the best case is the absence of damping at all, the presence of even a small amount of damping significantly increases the phase dynamic error. At the same time, increasing the damping causes the error to approach the ideal case.

Let us now study the amplitude dynamic error. Substituting the amplitude (13) into the (14) gives the expression for the amplitude dynamic error:

$$\begin{aligned} E_A(\delta\lambda) = \{&(\delta\lambda + g_2\delta\omega)[(\delta k^2 - \delta\omega^2)^2 + 4\delta k^2\zeta^2\delta\zeta^2\delta\omega^2]^{1/2} \times \\ &\times[(1-\delta\omega^2)^2 + 4\zeta^2\delta\omega^2]^{1/2}\}/\{g_2\delta\omega[(\delta k^2 - (\delta\lambda+\delta\omega)^2)^2 + \\ &+ 4\delta k^2\zeta^2\delta\zeta^2(\delta\lambda+\delta\omega)^2]^{1/2}[(1-(\delta\lambda+\delta\omega)^2)^2 + 4\zeta^2(\delta\lambda+\delta\omega)^2]^{1/2}\} \end{aligned} . \tag{15}$$

Note that amplitude dynamic error, given by (15), doesn't explicitly depend on the primary natural frequency. Maximisation of the CVG sensitivity requires small natural frequency of the primary oscillations, due to the k^3 term in the denominator of the amplitude (13). Providing the necessary bandwidth of the sensor requires keeping the amplitude dynamic as low as possible (namely 1) within that bandwidth.

Graphic plot of the amplitude dynamic error as function of the primary oscillations damping coefficient ζ and relative frequency of the angular rate is shown in the figure 2. From this graph one could see, that decreased damping results in significant dynamic error even for the small frequencies of the angular rate. On the other hand, increased damping causes the drop between peaks to diminish, thus providing low values for the dynamic error.

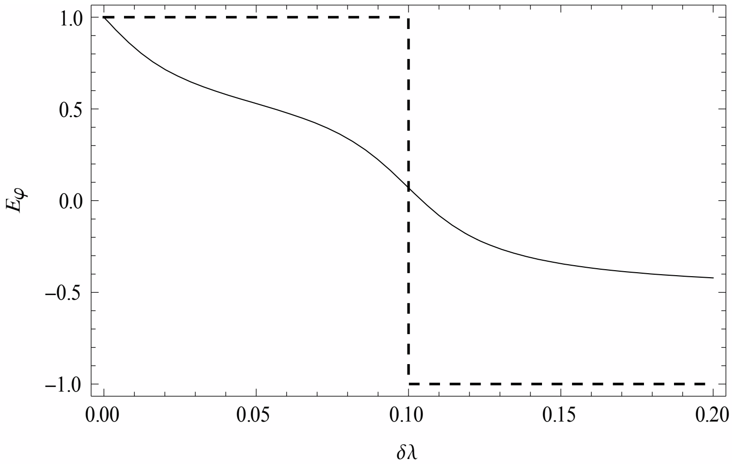

Fig. 1. Phase dynamic error
(solid line - $\zeta = 0.02$, dashed line - $\zeta = 0$, $\delta k = 1.1$, $\delta\omega = 1, \delta\zeta = 1$)

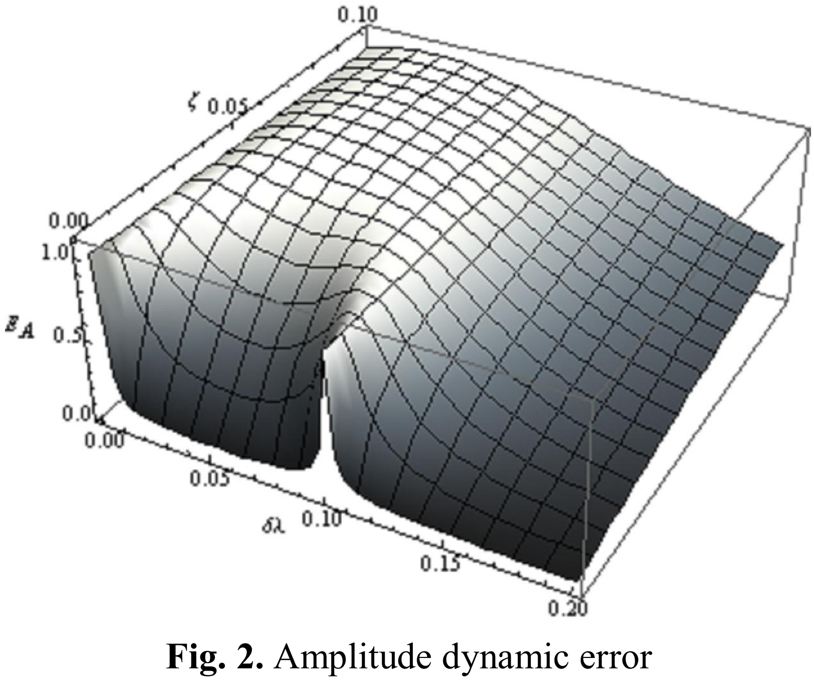

Fig. 2. Amplitude dynamic error
($\delta k = 1.1$, $\delta\omega = 1, \delta\zeta = 1$)

Amplitude dynamic error has two maximums and one local minimum along the rate frequency axis that are clearly visible on the figure 2, especially in the case of low damping. Positions of these extremums can be found from the following equation:

$$\frac{d}{d(\delta\lambda)} E_A(\delta\lambda) = 0. \tag{16}$$

General solution of the equation (16) is quite difficult to analyse. However, in case of zero damping it can be significantly simplified and its solutions could be found from the following equation:

$$(\delta\lambda+\delta\omega)[1+\delta k^2-2(\delta\lambda+\delta\omega)^2][\delta k^2-(\delta\lambda+\delta\omega)^2][(\delta\lambda+\delta\omega)^2-1]=0. \quad (17)$$

Three positive roots of the equation (17) are

$$\delta\lambda_1=1-\delta\omega,\ \delta\lambda_2=\sqrt{\frac{1+\delta k^2}{2}}-\delta\omega,\ \delta\lambda_3=\delta k-\delta\omega. \quad (18)$$

Here the first and the last roots correspond to maximums, and the second one to the minimum.

In general terms, optimization of the bandwidth means providing the same zero level of the amplitude dynamic error at each of three extremums in vicinity of the given by (18) frequencies of the angular rate. Amplitude dynamic error level at the second maximum, which corresponds to $\delta\lambda_3$, can be controlled by the damping ratio $\delta\zeta$ if it is a root of the following equation:

$$E_A(\delta\lambda_3)=1. \quad (19)$$

Positive solution of the equation (19) assuming $\delta\omega=1$ is

$$\begin{aligned}\delta\zeta=(\delta k^2-1)(\delta k+g_2-1)/\delta k[g_2^2(\delta k^2+\delta k^6-4\zeta^2+2\delta k^4(2\zeta^2-1))-\\-4\zeta^2(\delta k-1)^2-8g_2\zeta^2(\delta k-1)]^{1/2}\end{aligned}. \quad (20)$$

Now let us find the damping ζ that satisfies the equation

$$E_A(\delta\lambda_2)=1, \quad (21)$$

where $\delta\zeta$ is given by the expression (20). In this case equation (21) will include only depend on natural frequencies ratio and unknown damping ζ. Full expression for the equation (21) is quite large to be shown here, however it could be easily solved numerically. Graphic plot of the amplitude dynamic error as a function of damping ζ and with respect to the optimal damping ratio (20) is shown in the figure 3.

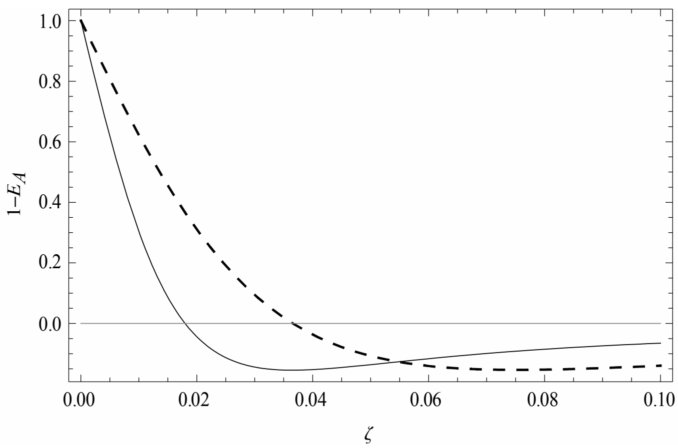

Fig. 3. Minimal dynamic error damping
(solid line - $\delta k=1.05$, dashed line - $\delta k=1.1$)

For example, optimal damping parameters for $\delta k = 1.05$, are $\zeta = 0.018$, $\delta\zeta = 0.921$. Amplitude dynamic error for this case is shown in the figure 4.

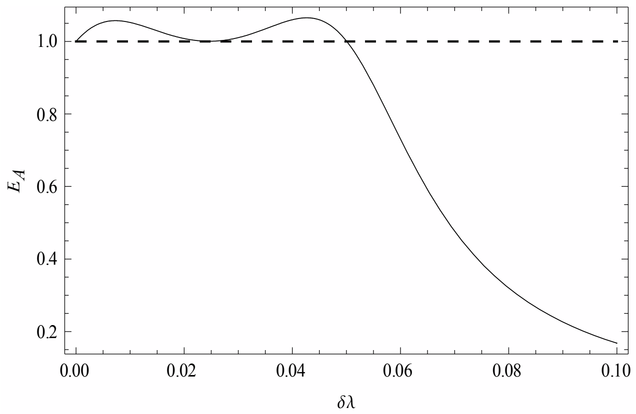

Fig. 4. Optimized amplitude dynamic error

One should also note that achieved level of the amplitude dynamic error could further improved if the objective level of the dynamic error in equations (19) and (21) is set to $1-e$ instead of 1, where e is the acceptable value of the dynamic error.

Bandwidth realization

Based on the presented above analysis of the dynamics errors of the CVGs, necessary bandwidth can be achieved by means of proper choice not only of the natural frequencies ratio, as was suggested in [5], but by providing proper damping of the primary and secondary oscillations as well.

In order to provide necessary bandwidth, natural frequencies ratio could be chosen based on the position of the second maximum in the amplitude response:

$$\delta k = \delta\lambda_* + \delta\omega \,. \tag{22}$$

Here $\delta\lambda_*$ is the required bandwidth in the dimensionless form, related to the natural frequency of primary oscillations. After the natural frequency ratio is calculated using (22), the result is used to calculate necessary damping for the primary oscillations. The latter problem can be solved either numerically or even analytically for some simplified cases. Having now calculated proper frequency ratio and primary damping, corresponding secondary damping is calculated using the ratio (20).

Considering the fact, that providing necessary damping in the CVGs is not an easy task, optimal values can be implemented using closed-loop operation both for the primary and secondary oscillations.

Conclusions

From the presented above study of dynamics errors of the Coriolis vibratory gyroscopes the following conclusions can be made:

- Using amplitude-phase variables instead of generalized coordinates allows obtaining proper transfer functions, where input is the measured angular rate, rather than actuation.
- Using suggested above procedure, CVG bandwidth can be realized more efficiently by tuning not only the frequency ratio, but damping as well.

Further study is required to devise algorithms for the closed-loop operation in order to ensure damping optimal in terms of bandwidth realization.

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DYNAMIC ERRORS OF CORIOLIS VIBRATORY GYROSCOPES

Analysis of the Coriolis vibratory gyroscopes sensitive element dynamics in terms of the amplitude-phase variables leading to the proper transfer functions of such inertial sensors is proposed in this paper. Obtained transfer function is used to derive expressions for the amplitude and phase response of such sensors, which in turn allows proper analysis of its dynamic errors. Based on the dynamic errors analysis, bandwidth realization method for Coriolis vibratory gyroscopes is presented as well.

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ДИНАМІЧНІ ПОХИБКИ КОРІОЛІСОВИХ ВІБРАЦІЙНИХ ГІРОСКОПІВ

В цій статті представлено аналіз динаміки чутливого елемента Коріолісових вібраційних гіроскопів в амплітудно-фазових змінних, що веде до виведення корректної передатної функції таких інерціальних датчиків. Отримана передатна функція використана для виведення виразів для амплітудно- та фазо-частотних характеристик датчиків, що в свою чергу дозволили виконати аналіз їх динамічних похибок. На основі представленого аналізу динамічних похибок було отримано метод забезпечення полоси пропускання для Коріолісових вібраційних гіроскопів.

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ДИНАМИЧЕСКИЕ ПОГРЕШНОСТИ КОРИОЛИСОВЫХ ВИБРАЦИОННЫХ ГИРОСКОПОВ

В этой статье представлен анализ динамики чувствительного элемента Кориолисовых вибрационных гироскопов в амплитудно-фазовых переменных, который позволяет получить корректную передаточную функцию таких инерциальных датчиков. Полученная передаточная функция использована для получения выражений для амплитудно- и фазо-частотных характеристик датчиков, что в свою очередь позволило выполнить анализ их динамических погрешностей. На основе представленного анализа динамических погрешностей был получен метод обеспечения полосы пропускания для Кориолисовых вибрационных гироскопов.